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Study of Some Prominent Fractional Type of Derivatives and Integrals, and Solution by Sumudu Transform Method to Ordinary Differential Equations of Fractional Order

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ABSTRACT: There are various numerical and analytical methods available for solving the fractional type ordinary differential equations. Applying the Sumudu integral transform, these equations can be solved with more accuracy. This method can be applied to solve differential equations of fractional and non-fractional,homogeneous and non-homogeneous type.

Keywords: Sumudu integral transform; fractional derivatives; fractional integrals and differential equations.

INTRODUCTION: Sumudu transform method, for solving fractional type ordinary differential equations was established by Belgacem and carried further for more contributions in the field by Kilckman and others working in this area. Adomain decomposition method was George Adomain; it is one of the very potential methods for analyzing such differential equations. In this paper objective is to present more specific and appropriate solutions to differential equations fractional order.

Many advance studies in solid state physics, control systems, signal processing, thermodynamics, and stochastic processes involve fractional type ordinary differential equations and applications. Various methods are instrumental till date for solving the fractional type ordinary differential equations which includes Adomain decomposition, Homotopy decomposition originated by expert researchers in this field **About the Sumudu Transform:**

Sumudu transform is defined as

$$F(s) = S[f(s)] = \int_0^{\infty} f(ut)e^{-t}\,dt, u \in \{-\tau_1, \tau_2\}$$

Over the set of functions given by

$$A = \{ f(t) \mid \ni M,$$

$$\tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{|t|}{\tau_1}}, if\ t \in (-1)^j \times [0, \infty)\}$$

Two basic types of Fractional order derivatives: Fractional calculus is a widely used mathematical concept for applied sciences. Nevertheless, it is somehow very difficult to deal and merely in last few years researchers have been encouraged to use the associated concepts. Various types of fractional order integrals were introduced by Poldubny for a simple interpretation of concept. He proposed a useful real field and material explanation of fractional integral in terms of non-homogeneous and changing time scale. Some of the prominent fractional derivatives and integrals with their inherent characteristics are expressed as follows.

There are plenty of ways to deal with fractional derivatives, following are few of them:

(i) Riemann-Liouville fractional derivative and fractional integral
(ii) Caputo Fractional derivative
(iii) Jumarie's Fractional derivative

We review some concepts of derivatives and integrals which are useful for further research in fractional calculus.

Definitions:

(i) Riemann-Liouville fractional integral

$$If\ f(x) \in C[a, b] and\ a < x < b,$$

Riemann-Liouville fractional integral of f (X)is denoted by $I^{\beta}_{\alpha+}f(x)$ and is defined as,

$$I^{\beta}_{\alpha+}f(x) = \frac{1}{\gamma(\beta)} \int_a^x \frac{f(t)}{(x-t)^{1-\beta}}\,dt, \beta \in (-\infty, \infty)$$

And $D^{-\alpha}(f(t)) = \frac{1}{\gamma(\alpha)} \int_0^t (t-\tau)^{(\alpha-1)} f(\tau)d\tau.$
$0 < \alpha \leq 1;$

Abel-Riemann fractional integral is given by$J^{\propto}, J^{\propto}(f(t)) = \int_0^t (t-\tau)^{(\alpha-1)} f(\tau)d\tau, t \geq 0, \alpha > 0;$

Using this definition, $J^{\propto}$,
$J^{\propto}(t^n) = \frac{\gamma(1+n)}{\gamma(1+n+\alpha)} t^{(n+\alpha)}, and\ D^{\propto}(t^n) = \frac{\gamma(1+n)}{\gamma(1+n-\alpha)} t^{(n-\alpha)}$

(ii) Riemann- Liouville fractional derivative:

$If\ f(x) \in C^n[a,b] and\ a < x < b,$

The Rieman Liouville fractional derivative of f (x) is denoted by

$\alpha^{D_x^\beta} = \frac{1}{\gamma(n-\beta)} \frac{d^n}{dx^n} \int_a^x \frac{f(\tau)}{(x-\tau)^{(\beta-n+1)}} d\tau, n-1 < \beta < n$

Abel-Riemann fractional derivative is given by,

$$D^{\propto}(f(t)) = \{\frac{1}{\gamma[m-n]} \frac{d}{dt^m} \int_0^t \frac{f(t)}{(t-\tau)^{(\alpha-m+1)}}\ for\ m-1 < \alpha < m;$$

$$D^{\propto}(f(t)) = \frac{d^m}{dt^m} f(t), for \alpha = m, m \in Z^+, \alpha \in R^+\ and\ D^{\propto}\ is\ the\ derrivative\ operater$$

(iii) Fractional order derivative concept given by Caputo is as follows:

$If\ f(x) \in C^n[a,b] and\ a < x < b,$

Then Caputo fractional derivative of f (x) is denoted by

$\alpha^{D_x^\beta} = \frac{1}{\gamma(n-\beta)} \frac{d^n}{dx^n} \int_a^x \frac{f(\tau)}{(x-\tau)^{(\beta-n+1)}} d\tau, n-1 < \beta < n$

or

$C_{D^\alpha} f(t) = \frac{1}{\gamma(n-\alpha)} \int_0^t \frac{f^n(\tau)}{(t=\tau)^{\alpha-n+1}} d\tau, for\ n-1 < \alpha < n;$

$C_{D^\alpha} f(t) = \frac{d^n}{dt^n} f(t), \text{for } \alpha = n,$ and

$J^{\propto}(C_{D^\alpha} f(t)) = f(t) - \sum_0^\infty f^k\ (0^+) \frac{t^k}{k!}$.

(iv) Jumarie's Derivative: The Jumarie's Derivative of order

$\alpha \geq 0\ of\ a\ function\ f \in C_\mu, \mu \geq -1, is\ defined\ as,$

$$J^\alpha(f(x)) = \frac{1}{\gamma(m-\alpha)} \frac{d^m}{dx^m} \int_a^x (x-t)^{(m-\alpha)} [f(t) - f(0)] dt, \quad \alpha > 0, x > 0$$

If f (t) is inverse Sumudu Transform of F (u), then transform of integer order derivative of f(t) is given by ,

$$S\left[\frac{d^n}{dt^n} f(t)\right] = \frac{1}{u^n}[F(u) - \sum_0^{n-1} u^k \frac{d^n}{dt^n} f(t)|_{t=0}]$$

If f (t) is inverse Sumudu Transform of F (u), then transform of fractional order Riemann –Liouville derivative of f(t) is given by ,

$$S[D^\alpha f(t)] = u^{-\alpha}\left[F(u) - \sum_0^n u^{\alpha-k}[D^{\alpha-k} f(t)|_{t=0}\right], -1 < n-1 \leq \alpha < n$$

Sumudu transform of Caputo order fractional derivative is given by

$$S[D^\alpha f(t)] = \frac{S[f(t)]}{u^\alpha} - \sum_0^{n-1} \frac{f_{(0)}^k}{u^{(\alpha-k)}} . n-1 < \alpha < n$$

Now let us apply this concept to solve fractional type of a linear differential equation. In this part we apply STM. Consider the fractional order differential equation,

$D^{\frac{5}{2}}(y) + D^{\frac{3}{2}}(y) + D^{\frac{1}{2}}(y) = cosht$ With the initial conditions y (0) = 0, $y'(0) = 0, y''(0) = 0$

Taking Sumudu transform on both sides,

$[u^{\frac{5}{2}} + u^{\frac{3}{2}} + u^{\frac{1}{2}}]y(u) - u^{\frac{-5}{2}}y(0) - u^{\frac{-3}{2}}y'(0) - u^{\frac{-5}{2}}y''(0) - u^{\frac{-3}{2}}y(0) - u^{\frac{-1}{2}}$

$y'(0) - u^{\frac{-1}{2}}y(0) = \frac{1}{(1-u^2)}$

Simplifying, y (u) $= \frac{(1-u^2)}{[u^{\frac{5}{2}}+u^{\frac{3}{2}}+u^{\frac{1}{2}}]}$, Taking inverse Sumudu transform,

$Y(t) = \frac{-1}{6\sqrt{\pi}} e^{\frac{-t}{2}} \int_0^t \frac{e^{\frac{t}{2}}[\cos(\frac{\sqrt{3}}{2}(t-u)) - \sqrt{3\sin\frac{\sqrt{3}}{2}(t-u)}]}{\sqrt{u}} du - \frac{1}{3\sqrt{\pi}} e^{\frac{-t}{2}} \int_0^t \frac{e^{\frac{t}{2}}[\sin(\frac{\sqrt{3}}{2}(t-u))}{\sqrt{u}} du - \frac{1}{6} e^t \operatorname{erf}(\sqrt{t}) - \frac{-1}{6\sqrt{\pi}} e^{\frac{-3t}{2}} \int_0^t \frac{e^{\frac{t}{2}}[\cos(\frac{\sqrt{3}}{2}(t-u)) + \sqrt{3\sin\frac{\sqrt{3}}{2}(t-u)}]}{\sqrt{u}} du + \frac{-1}{3\sqrt{\pi}} e^{\frac{-3t}{2}} \int_0^t \frac{e^{\frac{t}{2}}[\sin(\frac{\sqrt{3}}{2}(t-u))}{\sqrt{u}} du - i\frac{1}{2} e^{-t} \operatorname{erf}(i\sqrt{t})$

Now we solve another fractional order differential equation,

$$\left[D + D^{\frac{1}{2}} - 2\right] y(t) = 0, subjected\ to\ D^{\frac{1}{2}} y(0^+) = c, y(0^+) = 0, t > 0$$

Taking Sumudu transform on both sides,

$$S\left[D + D^{\frac{1}{2}} - 2\right] y(t) = S(0)$$

Implies,

$\frac{1}{u}[F(u) - f(0)] + \frac{F(u)}{u^{\frac{1}{2}}} - \frac{D^{\frac{-1}{2}} f(t)}{u}|_{t=0} - 2F(u) = 0 \Rightarrow$

$\frac{F(u)}{u} + \frac{F(u)}{\sqrt{u}} - \frac{c}{u} - 2F(u) = 0 \Rightarrow$ $(1 + u^{\frac{1}{2}} - 2u)F(u) = c \Rightarrow$

F (u) $= \frac{c}{(1+u^{\frac{1}{2}}-2u)} \Rightarrow \frac{2c}{3(1+2\sqrt{u})} + \frac{c}{3(1-\sqrt{u})}$ =>taking inverse sumudu transform

F(u)$= \frac{2c}{3}\left[\frac{(1-2\sqrt{u})}{(1-4u)}\right] + \frac{c}{3}[\frac{(1+\sqrt{u})}{(1-u)}] \Rightarrow$ f(t)$= \frac{2c}{3}[e^{4t} -$ *e4t*erf2*t*+*c*3*[et*+*et*erf *(t)]*

This solution by Sumudu transform method is novel and does not resemble with any solution obtained by other methods.

CONCLUSION: The prime concept of this paper is to establish the novel method of finding solution to differential equations fractional and non-fractional order. In this paper we solved different types of fractional order differential equations by Sumudu transform method. We could obtain appreciable perfect solutions for the numerical. This shows efficiency of this method in obtaining accurate solutions to ordinary fractional differential equations. This method is effective and successful for solving the critical fractional order differential equations. The prominent utility of this method is obtaining analytical method. It is seen that this method is an inspiring technique for solving fractional differential equations.

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